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Optimized functional perturbation method and morphology based effective properties of randomly heterogeneous beams

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Abstract

A Functional Perturbation Method (FPM) has been recently developed for the analysis of stochastically heterogeneous structures, for which the heterogeneity scale is not negligible relative to the macro dimensions. The FPM is based on considering the target function (here, the buckling load P) as a functional of the stochastic morphology. The target function is written as a functional perturbation series near a convenient homogeneous property, usually stiffness (K) or compliance (S). Thus, the accuracy depends on the choice of the property around which the perturbation is carried out. An Optimized FPM (OFPM) is presented here, which concentrates on finding a property $\theta(K)$, which is a function of K or S, such that the target function converges faster. This is accomplished by looking for $\theta(K)$ which minimizes (or nulls, if possible) the second term in the functional perturbation series. Besides its improved accuracy, θ has also a dual meaning, which is related to the notion of "effective" property. However, the "effectiveness" is weak, since the property is not "purely material", but depends on external loading shapes. An example of a buckling problem is examined in detail, for which θ is found analytically as a simple power of K, which directly depends on morphology. Comparing the new OFPM with previous FPM and numerical Monte Carlo—Finite Element results shows the desired improved accuracy. The advantages of the OFPM are then shown and discussed.
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1. Introduction and motivation

In recent studies (Altus, 2001, 2003a,b; Givli and Altus, 2003), a new Functional Perturbation Method (FPM) has been developed, which predicts the mechanical probabilistic behavior of stochastically heterogeneous beams. This includes random properties such as deflections, buckling loads, beam's strength and fracture energy. Their statistical averages and variances have been found in terms of morphology, loading and geometry. These properties will be termed "target functions", for convenience.

The analysis of structures with stochastic material properties has been a major subject of research for more than three decades. Much of the effort is devoted to implementing the randomness into the Finite Element Method (SFEM) (Ghanem and Spanos, 1991; Deodatis, 1991; Graham and Siragy, 2001; Elishakoff and Ren, 2003; Frantziskonis and Breysse, 2003, to name a few). In all of the existing methods, the random properties are parametrically dependent, i.e., the stochastic nature of the problem is transformed to an associated structure with finite number of random variables.

The FPM is based on treating the target functions as *functionals* of material heterogeneity (morphology). Then, they are functionally expanded as a Fréchet series around an arbitrarily chosen uniform (associated with a homogeneous) material property. Therefore, the average of the target functions for the heterogeneous case is obtained as a series in which the first term (zero perturbation order) is the solution of a homogeneous structure having material property which is equal to the average value of the chosen property for the heterogeneous case.

Although the method is analytical, the accuracy of the solution is affected by three different sources of approximations: (a) The beam deflection shape functions which are chosen to cover the maximum possible solution subspace (i.e., Bernoulli, Timoshenko or any other high order theories), (b) The number of terms used in the functional perturbation series, and (c) The specific homogeneous material property, around which the FPM is executed. While the first two sources come from "common" accuracy related problems, which are treated extensively in the literature, the third involves functional differentiations and is directly related to the FPM itself. Thus, the third source is the prime subject of this study. It will also be shown how the last two subjects are interrelated.

A good demonstration of the importance of choosing the proper homogeneous property is through the solution of the buckling problem (Altus and Totry, 2003b). Similar to other beam problems, it is "natural" to execute the functional perturbation either around the average cross section stiffness $\langle K\rangle$ or compliance $\langle S\rangle$ of the beam. However, examining the two limit morphological cases, associated with very large or very small moduli correlation lengths (roughly, the typical grain size), it is easy to show that $\langle K\rangle$ and $\langle S\rangle$ are the values giving the *exact* average target function (buckling load) for infinitely large and infinitely small grains, respectively. Thus, it is clear that for a finite grain size the exact property *must* be neither of the two but some *function* (θ) of S or K and is morphology dependent. Finding this function by an Optimized Functional Perturbation Method (OFPM) is the main goal of this study. To the best of our knowledge, this type of optimization has not been considered before. The reason may be that effective properties are practically related to infinite bodies (microstructure sizes much smaller than global dimensions).

Apart from its improved accuracy, the OFPM serves a dual goal, which is even more fundamental, and is related to the notion of *effective* material property, such as moduli. By definition, the value of the target function of a structure made from a homogeneous material with the effective property is equal to the value of the target function of the same structure made from a heterogeneous material. However, this "classical" definition is for heterogeneity on a very small scale, i.e., when grain size is negligible compared to the macro dimensions. In this case, the target function is practically not random. However, in the present study the grains are not small and the definition of effective property is naturally associated with the *average* target function of the heterogeneous structure.

Returning to the series solution of the target function, in which the first term is the solution for a corresponding homogeneous case, i.e.,

$$\langle P\{S(x)\}\rangle = P_{\langle\theta(S)\rangle} + \cdots \tag{1.1}$$

It is seen that the average of the chosen material property (θ) can be also considered as a first approximation for an effective property. Thus, the OFPM also serves to find a better estimate for the effective moduli of heterogeneous structures with finite grain size which may be very useful for design of stochastically heterogeneous structures. Note however, that contrary to the infinite body case, this is an effective property which will be dependent on *morphology and loading geometry*. It is also realized, that $\langle K\rangle$, $\langle S\rangle$ or $\langle\theta(K)\rangle$ are all one point probability functions which can be found from morphological data with the same effort.

2. Theoretical considerations

Consider a mechanical structure, for which it is desired to find a property P (say, a critical load of some kind). The structure is made from a randomly heterogeneous material of property $S(x)$, which is statistically stationary. Let us consider a one dimensional case for easy insight.

Assume that P can be written as an explicit functional of S. To find the statistical characteristics of P, the Functional Perturbation Method (FPM) is used (Altus and Givli, 2003a; Altus and Totry, 2003b). A Fréchet expansion around the ensemble average of S is written as:

$$P\{S(x)\} = P_{\langle S\rangle} + P_{,S_1} * S_1' + \frac{1}{2}P_{,S_1S_2} * *S_1'S_2' + \cdots, \tag{2.1}$$

($*$) is the common convolution symbol, { } is for a functional relation and all functional derivatives are taken at $\langle S\rangle$. Also,

$$S' = S - \langle S\rangle; \quad P_{,S} = \frac{\delta P}{\delta S(x)}; \quad P_{,S_1} = \frac{\delta P}{\delta S(x_1)}, \tag{2.2}$$

etc. The statistical (ensemble) average of P is then:

$$\langle P\rangle = P_{\langle S\rangle} + \frac{1}{2}P_{,S_1S_2} * *\langle S_1'S_2'\rangle + \cdots \tag{2.3}$$

Each additional term in the series contains more detailed information about the stochastic properties of $S(x)$. For example, $\langle S_1'S_2'\rangle$ is the two point probability function (2pp) which usually contains a typical "grain size" measure λ, defined by:

$$\langle S_1'S_2'\rangle * 1 = 2\lambda\langle S_1'^2\rangle. \tag{2.4}$$

The above is not unique, and other "sizes" can be defined. However, all definitions must have the common property, that when the microstructure is self similarly enlarged by an arbitrary factor, λ should also be changed by the same factor.

Other statistical properties of P, such as Var(P), can be found similarly:

$$\langle P'^2\rangle = P_{,S_1} * \langle S_1' \cdot S_2'\rangle * P_{,S_2} + \cdots \tag{2.5}$$

When P is not given explicitly, the functional derivatives of P in (2.1) are found successively as follows. Assume an implicit relation of the form

$$\Phi(P\{S(x)\}) = 0. \tag{2.6}$$

Apply the zero order (homogeneous) substitution:

$$\Phi(P\{\langle S\rangle\}) = 0 \tag{2.7}$$

From which $P(\langle S\rangle)$ is found. Then, differentiate (2.6) functionally to obtain:

$$\Phi_{,P}P_{,S_1}|_{\langle S\rangle}=0. \tag{2.8}$$

Since $\Phi_{,P}$ (ordinary derivative) at $\langle S\rangle$ is a known function of $P(\langle S\rangle)$, we can use the result from (2.7), substitute in (2.8) and find $P_{,S_1}(\langle S\rangle)$ (functional derivative). Differentiating (2.8) again

$$(\Phi_{,PP}P_{,S_1}P_{,S_2}+\Phi_{,P}P_{,S_1S_2})_{\langle S\rangle}=0. \tag{2.9}$$

Using the results of (2.7) and (2.8), $P_{,S_1S_2}(\langle S\rangle)$ can be found. Continuing in this manner, it is straightforward to find any desired functional derivative of P at $\langle S\rangle$.

The function $S(x)$, which is taken here as material moduli or compliance of elastic heterogeneous structure, is the one which regularly appears in the original formulation (governing equations) of the non-random problem. However, the choice of functionally expanding through S or any other function $\theta(S)$ is still at our disposal. This is our basis for the OFPM.

2.1. Optimization for improved accuracy

Consider a practical case when morphological data is given up to the two point moduli probability, therefore only two terms of (2.3) are completely available. Our aim is to find $\theta(S)$ such that, when expanded as (2.3) by θ, the magnitude of the second term is minimal, or, as will be shown later in the example, causes the second term to vanish. Then, if the functional series converges monotonically, the first two terms are expected to yield the most accurate value of $\langle P\rangle$ for this given (limited) morphology data. Expansion of P through θ yields:

$$\langle P\rangle=P_{\langle\theta\rangle}+\frac{1}{2}P_{,\theta_1\theta_2}**\langle\theta_1'\theta_2'\rangle+\cdots\cong P_{\langle\theta\rangle}+\frac{1}{2}P_{,\theta_1\theta_2}**\langle\theta_1'\theta_2'\rangle. \tag{2.10}$$

We ask for the property $\theta(S)$ such that

$$P_{,\theta_1\theta_2}**\langle\theta_1'\theta_2'\rangle=0. \tag{2.11}$$

If there is no solution for (2.11), we will ask for a minimization condition. To write (2.11) in terms of S, the following functional differentiations are needed:

$$P=P\{S(\theta(x))\}\to P_{,\theta_1}=P_{,S_1}S_{1,\theta_1} \tag{2.12}$$

and

$$P_{,\theta_1\theta_2}=P_{,S_1\theta_2}S_{1,\theta_1}+P_{,S_1}S_{1,\theta_1\theta_2}=P_{,S_1S_2}S_{2,\theta_2}S_{1,\theta_1}+P_{,S_1}S_{1,\theta_1\theta_1}\delta_{12}. \tag{2.13}$$

Note that $S_{,\theta}$ is a regular derivative, since S is taken here as a function (and not functional) of θ. The following relation and notation has been used in the above

$$S_{1,\theta_1\theta_2}=S_{1,\theta_1\theta_1}\delta_{12};\quad \delta_{12}\equiv\delta_{x_1x_2}\equiv\delta(x_1-x_2), \tag{2.14}$$

where δ is the Dirac singular operator, defined as

$$\delta_{12}=\frac{\delta S(x_1)}{\delta S(x_2)}\equiv S_{1,S_2}=\frac{\delta\theta(x_1)}{\delta\theta(x_2)}\equiv\theta_{1,\theta_2}. \tag{2.15}$$

Substitution of (2.13) in (2.11) we note that for a statistically homogeneous distribution

$$(P_{,S_1}S_{1,\theta_1\theta_1}\delta_{12})**\langle\theta_1'\theta_2'\rangle=(P_{,S_1}S_{1,\theta_1\theta_1})*\langle\theta_1'^2\rangle=\langle\theta'^2\rangle\cdot(P_{,S}S_{1,\theta\theta})*1. \tag{2.16}$$

The condition (2.11) is thus

$$P_{,\theta_1\theta_2}**\langle\theta_1'\theta_2'\rangle=(P_{,S_1S_2}S_{2,\theta_2}S_{1,\theta_1})**\langle\theta_1'\theta_2'\rangle+(P_{,S_1}S_{1,\theta_1\theta_1})*\langle\theta_1'^2\rangle=0. \tag{2.17}$$

$1_1 \equiv 1(x_1)$ is a uniform function of unit strength along the x_1 coordinate. A normalized two point probability correlation of θ is defined by

$$\langle \bar{\theta}'_1 \bar{\theta}'_2 \rangle = \frac{\langle \theta'_1 \theta'_2 \rangle}{\langle \theta'^2_1 \rangle}. \tag{2.18}$$

Then, (2.17) is written as

$$(P_{,S_1S_2} S_{2,\theta_2} S_{1,\theta_1}) ** \langle \bar{\theta}'_1 \bar{\theta}'_2 \rangle + (P_{,S_1} S_{1,\theta_1\theta_1}) * 1_1 = 0. \tag{2.19}$$

To write (2.19) as a net differential equation with $S(\theta)$ as unknown, we expand

$$\theta' = \theta - \langle \theta \rangle = \left(\theta_{\langle S \rangle} + \theta_{,S} S' + \frac{1}{2} \theta_{,SS} S'^2 + \cdots \right) - \left(\theta_{\langle S \rangle} + \frac{1}{2} \theta_{,SS} \langle S'^2 \rangle + \cdots \right)$$

$$= \theta_{,S} S' + \frac{1}{2} \theta_{,SS} (S'^2 - \langle S'^2 \rangle) + \cdots. \tag{2.20}$$

Then

$$\langle \theta'_1 \theta'_2 \rangle = \theta_{,S_1} \langle S'_1 S'_2 \rangle \theta_{,S_2} + O(S'^3). \tag{2.21}$$

Since (regular) derivatives with respect to S are taken at $\langle S \rangle$,

$$\theta_{,S_1} = \theta_{,S_2} = \theta_{,S} \tag{2.22}$$

are all constants, and *not* functions of space. Therefore, substituting (2.21) in (2.19) and using (2.22) we obtain:

$$(P_{,S_1S_2} S_{2,\theta_2} S_{1,\theta_1}) ** \langle \bar{S}'_1 \bar{S}'_2 \rangle + (P_{,S_1} S_{1,\theta_1\theta_1}) * 1_1 = O\langle \bar{S}'^3 \rangle \cong 0. \tag{2.23}$$

Since the whole calculation is confined to the second order analysis, neglecting the third order term will not affect the overall accuracy. The above constitutes a non-linear differential equation for $S(\theta)$ for *any* morphology. Nevertheless, it is interesting to consider further a less general case, for which the two-point probabilities can be decoupled into two parts: "micro-material" and "micro-geometry", in the form:

$$\langle S'_1 S'_2 \rangle = \langle S'^2_1 \rangle \cdot p^{(2)}(| x_2 - x_1 |). \tag{2.24}$$

A common example for this case is when morphology is based on grains with uniform moduli inside each grain but exhibits no correlation between them. There, $p^{(2)}$ is the probability that two points will "fall" in the same grain, which is a pure geometrical property (Kröner, 1986). It is seen from (2.18) and (2.24) that in such cases, the normalized 2pp's of θ and S are identical:

$$\langle \bar{\theta}'_1 \bar{\theta}'_2 \rangle = \langle \bar{S}'_1 \bar{S}'_2 \rangle. \tag{2.25}$$

Then, for this particular case, (2.23) is *exactly* zero.

Recall that since all functional derivatives in (2.23) are at $\langle \theta \rangle$, expressions such as $S_{1,\theta_1\theta_1}$ and S_{1,θ_1} are *not* functions of space, and the decoupling of the form

$$(P_{,S_1})_{\langle \theta \rangle} = a(x_1) \cdot A(S\langle \theta \rangle); \quad (P_{,S_1S_2})_{\langle \theta \rangle} = b(x_1, x_2) \cdot B(S\langle \theta \rangle) \tag{2.26}$$

is permitted. Using the above, (2.23) can be written in the following form, after space integrations:

$$[g \cdot G_{,S}(S_\theta) \cdot (S_{\theta,\theta})^2 + G(S_\theta) \cdot S_{\theta,\theta\theta}]_{\langle \theta \rangle} = 0. \tag{2.27}$$

G is a known function of $S(\theta)$ and g is a morphology dependent integration coefficient which can be calculated for any specific case. Dividing (2.27) by $S_{,\theta}$ and noting that

$$\frac{S_{\theta,\theta\theta}}{S_{\theta,\theta}} = [\mathrm{Ln}(S_{\theta,\theta})]_{,\theta}; \quad G_{,S}S_{\theta} = G_{,\theta}, \tag{2.28}$$

we can integrate by θ and obtain the simple relation:

$$S_{\theta,\theta} = C \cdot G^{-g}. \tag{2.29}$$

C is a constant of integration. Note that $G(S)$ depends on the type of problem, while g is a pure functional of morphology.

Using the stationary property and the averaging definition, it is clear that the function which relates $S(x)$ to $\theta(x)$ is identical to the one which relates $\langle S\rangle$ to $\langle\theta\rangle$:

$$S_{\langle\theta\rangle} = \widetilde{F}[\langle\theta\rangle] \to S_{\theta} = \widetilde{F}[\theta]. \tag{2.30}$$

Therefore, the solution of (2.30) provides us with $S(\theta)$ too.

Knowing $\widetilde{F}$ and having the statistical data of S, we have:

$$S = \widetilde{F}(\theta) \to \theta = \widetilde{F}^{-1}(S). \tag{2.31}$$

Then

$$\langle\theta\rangle = \langle\widetilde{F}^{-1}(S)\rangle \to S(\langle\theta\rangle) = \widetilde{F}\langle\widetilde{F}^{-1}(S)\rangle. \tag{2.32}$$

Knowing $\widetilde{F}$ from (2.27) and using (2.31), the desired new parameter is obtained. Alternatively, (2.32b) can be used directly for calculating the first term of (2.10) and obtain an optimized estimation for $\langle P\rangle$.

2.2. Morphology based effective moduli

Traditionally, effective moduli is a property related to the response of a homogeneous material (and/or a structure) which, when used for a corresponding heterogeneous problem, yields the same response. Usually, it is related to the case where the microscale (grain size etc.) is much smaller than any relevant macro dimension, and therefore the macro response is *not* random. Denote this effective property by $\theta^{(\mathrm{eff})}$, we have

$$P = P_{\langle S\rangle} + \frac{1}{2}P_{,S_1S_2} * *\langle S'_1 \cdot S'_2\rangle + \cdots = P_{S(\theta^{(\mathrm{eff})})}. \tag{2.33}$$

In our case, the microscale is not negligibly small, and it is natural to generalize the concept of "effectiveness" to the *average* property (here P), i.e.,

$$\langle P\rangle = P_{S(\theta^{(\mathrm{eff})})}. \tag{2.34}$$

Thus, by the two expansions (2.3) and (2.10), $\langle S\rangle$ and $\langle\theta\rangle$ can be considered as two approximations for the effective property:

$$\langle P\rangle \cong P_{\langle S\rangle} \cong P_{S(\langle\theta\rangle)} \cong P_{S(\theta^{(\mathrm{eff})})}. \tag{2.35}$$

We see that $\langle S\rangle$ can be considered as a zero order approximation for the effective moduli while $\langle\theta\rangle$, for which the second term in (2.10) vanishes, is a more accurate, second order estimate. From (2.27) and (2.30), $\langle\theta\rangle$ depends on grain size *and* loading, and therefore it is *not* a "pure" material property in the classical sense. Continuing the above method for higher order approximations (i.e. finding the proper moduli for which any number of desired terms vanishes simultaneously) is a more complex problem which is under current study and will not be discussed here.

In the next chapter, a demonstrating example of a buckling problem is solved by the OFPM in details.

3. Buckling of stochastically heterogeneous beams by the FPM

Consider a stochastically heterogeneous beam under compression. Heterogeneity is confined to the longitudinal direction only. A special feature of the problem is the fact that the exact average buckling load for very small or very large grains is based on $\langle S\rangle$ or $\langle K\rangle$ respectively, and it is clear that different functions of these properties should be used for finite grain sizes. Buckling problem have been studied by Garrett (1992), Zhang and Ellingwood (1995) and Elishakoff (2000), which used the stiffness $EI = S^{-1}$ as the reference material property for all morphology range. Recently, Altus and Totry (2003b) studied the effect of using each of the two on the accuracy of the solution. However, an optimized function, which is based on morphology, has not been considered yet.

A brief summary of the solution procedure and final results of the buckling problem by the FPM is given herein. More details can be found elsewhere (Altus and Totry, 2003b), although in a less general form. This chapter is therefore a modification and is limited to the minimum exposure which will be needed later for the OFPM analysis.

Let a heterogeneous beam be of length L, simply supported at both ends, and loaded by a compression force P. $E(x)$ and $I(x)$ are the non-uniform modulus and inertia, respectively. The bending stiffness (K) and compliance (S) are:

$$K(x) = EI(x); \quad S(x) = (K(x))^{-1}. \tag{3.1}$$

x is the normalized (by L) longitudinal coordinate, $(0 < x < 1)$.

Assume a buckling deflection of the form:

$$w(x) = A^{(1)}w^{(1)}(x) + A^{(2)}w^{(2)}(x) = A^{(1)}\mathrm{Sin}(\pi x) + A^{(2)}\mathrm{Sin}(2\pi x) \tag{3.2}$$

where $A^{(1)}$ and $A^{(2)}$ are functionals of morphology. $w^{(1)}(x)$ is the exact shape function for the homogeneous case and $w^{(2)}$ permits a non-symmetric part corresponding to the specific inhomogeneity of each beam. It should be noted that (a) the chosen wavelength of $w^{(2)}$ was shown to provide the best accuracy (Altus and Totry, 2003b), and (b) $A^{(1)}$ is a functional of heterogeneity too. More terms can be added for better accuracy.

The internal bending moment distribution $M(x)$ and the elastic energy U are:

$$M(x) = Pw(x); \quad U = \frac{1}{2}M^2 * S. \tag{3.3}$$

Applying the minimum potential energy principle, the condition for the onset of instability yields:

$$P\begin{bmatrix} PS * (w^{(1)})^2 - \pi^2/2 & PS * (w^{(1)}w^{(2)}) \\ PS * (w^{(1)}w^{(2)}) & PS * (w^{(2)})^2 - 2\pi^2 \end{bmatrix}\begin{Bmatrix} A^{(1)} \\ A^{(2)} \end{Bmatrix} = 0. \tag{3.4}$$

A solution of (3.4) yields a quadratic characteristic equation for the buckling load P:

$$\begin{aligned}\Phi(P) = {} & P^2((S * (w^{(1)})^2)(S * (w^{(2)})^2) - S * (w^{(1)}w^{(2)})^2) \\ & - P\frac{\pi^2}{2}(4S * (w^{(1)})^2 + S * (w^{(2)})^2) + \pi^4 = 0,\end{aligned} \tag{3.5}$$

Eq. (3.5) is a functional equation for $P\{S(x)\}$ which is a realization of (2.6). We choose an uncoupled morphology of the exponential type (2.24):

$$\langle \overline{S}_1' \overline{S}_2' \rangle = \langle (\overline{S}')^2 \rangle \cdot \exp\left[-\frac{|x_2 - x_1|}{\lambda}\right]; \quad \overline{S}' = S'/\langle S \rangle. \tag{3.6}$$

Applying the FPM to (3.5) and (3.6) by using the procedure outlined in (2.6)–(2.9) yields:

$$\langle \overline{P}_S \rangle = \frac{\langle P_S \rangle}{P_{\langle S \rangle}} = 1 + \langle \overline{S}'^2 \rangle \cdot f(\lambda) = 1 + \rho_S^2 \cdot f(\lambda). \tag{3.7}$$

ρ_S is the Coefficient of Variation (COV) of S and

$$f(\lambda) = f^{(1)}(\lambda) - f^{(2)}(\lambda) \tag{3.8}$$

where,

$$f^{(1)}(\lambda) = \lambda \frac{3 + 20\pi^2\lambda^2 + 32\pi^4\lambda^4(1 - \lambda(1 - \exp(-\lambda^{-1})))}{(1 + 4\pi^2\lambda^2)^2}, \tag{3.9}$$

$$f^{(2)}(\lambda) = \lambda \frac{2 + 30\pi^2\lambda^2 + 118\pi^4\lambda^4 - 128\pi^4\lambda^5(\exp(-\lambda^{-1}) + 1) + 90\pi^6\lambda^6}{3(1 + 10\pi^2\lambda^2 + 9\pi^4\lambda^4)^2}. \tag{3.10}$$

$f^{(1)}$ and $f^{(2)}$ are contributions related to the two shapes of $A^{(1)}$ and $A^{(2)}$ in (3.2), respectively. The COV of $P(S)$ is obtained by (2.5)

$$\rho_{P(S)} = \frac{(\mathrm{Var}(P_S))^{1/2}}{\langle P_S \rangle} = \rho_S (f^{(1)})^{1/2}. \tag{3.11}$$

If the FPM is executed with K as a basis, we obtain:

$$\langle \overline{P}_K \rangle = \frac{\langle P_K \rangle}{P_{\langle K \rangle}} = 1 + \langle \overline{K}'^2 \rangle \cdot (f(\lambda) - 1) = 1 - \rho_K^2 \cdot (1 - f(\lambda)) \tag{3.12}$$

and

$$\rho_{P(K)} = \frac{(\mathrm{Var}(P_K))^{1/2}}{\langle P_K \rangle} = \rho_K (f^{(1)})^{1/2}. \tag{3.13}$$

It can be easily shown, that each of the two solutions is exact for a different limit case:

$$f(\lambda \to 0) = 0 \Rightarrow \langle P_S \rangle = P_{\langle S \rangle}; \quad \rho_{P(S)} = 0, \tag{3.14}$$

$$f(\lambda \to \infty) = 1 \Rightarrow \langle P_K \rangle = P_{\langle K \rangle}; \quad \rho_{P(K)} = \rho_K. \tag{3.15}$$

4. Implementing the optimized functional perturbation method (OFPM)

Assume that S is a function of the property $\theta(x)$, from which we obtain an average buckling load with improved accuracy:

$$S = S(\theta) = K(\theta)^{-1}. \tag{4.1}$$

Taking Φ from (3.5) and following the OFPM procedure outlined in chapter 2, the zero order equation for P is found by

$$\Phi(P\{S_{\langle\theta\rangle}\}) = 0. \tag{4.2}$$

We obtain two solutions of which only the smallest is relevant:

$$P^{(1)} = P_{S(\langle\theta\rangle)} = \frac{\pi^2}{S|_{\langle\theta\rangle}}; \quad P^{(2)} = \frac{4\pi^2}{S|_{\langle\theta\rangle}}. \tag{4.3}$$

Now find $P_{,S_1}$ from (2.8), insert in (2.12) and obtain (see Appendix for functional differentiation details):

$$P_{,\theta_1}|_{\langle\theta\rangle} = -2\pi^2 \frac{S_{,\theta}|_{\langle\theta\rangle}}{S|_{\langle\theta\rangle}{}^2}(w_{x_1}^{(1)})^2 \tag{4.4}$$

Similarly, use (2.9) to obtain $P_{,S_1S_2}$, insert in (2.13) and solve for $P_{,\theta_1\theta_2}$ using (4.3) and (4.4) to obtain:

$$P_{,\theta_1\theta_2}|_{\langle\theta\rangle} = \frac{\delta^2 P}{\delta\theta_1\delta\theta_2}|_{\langle\theta\rangle}\cdot = \frac{2\pi^2}{3S|^3_{\langle\theta\rangle}}\begin{bmatrix} S_{,\theta}|^2_{\langle\theta\rangle}(12(w_{x_1}^{(1)})^2(w_{x_2}^{(1)})^2 - 4(w_{x_1}^{(1)}w_{x_1}^{(2)})(w_{x_2}^{(1)}w_{x_2}^{(2)})) \\ -3S|_{\langle\theta\rangle}S_{,\theta\theta}|_{\langle\theta\rangle}(w_{x_1}^{(1)})^2\delta_{x_1x_2} \end{bmatrix}. \tag{4.5}$$

Note the abbreviations

$$(w_{x_1}^{(1)})^2 \equiv w_{x_1}^{(1)}w_{x_1}^{(1)} \equiv w^{(1)}(x_1)\cdot w^{(1)}(x_1) \tag{4.6}$$

etc., and

$$S_{,\theta}|_{\langle\theta\rangle} \equiv S_{,\theta(x)}|_{\langle\theta\rangle} \equiv S_{,\theta_1}|_{\langle\theta\rangle} \equiv S_{,\theta_2}|_{\langle\theta\rangle}. \tag{4.7}$$

Now insert the exponential 2pp from (3.6) into (4.5) and find $\theta(S)$ which nulls the second term:

$$\begin{aligned}&[S_{,\theta}|^2_{\langle\theta\rangle}\left(12(w_{x_1}^{(1)})^2(w_{x_2}^{(1)})^2 - 4(w_{x_1}^{(1)}w_{x_1}^{(2)})(w_{x_2}^{(1)}w_{x_2}^{(1)})\right) - 3S|_{\langle\theta\rangle}S_{,\theta\theta}|_{\langle\theta\rangle}(w_{x_1}^{(1)})^2\delta_{x_1x_2}] * *\langle\overline{\theta}'^2\rangle \\ &\quad\times\exp\left(-\frac{|x_2 - x_1|}{\lambda}\right) = 0.\end{aligned} \tag{4.8}$$

Integration over the space (x) yields

$$g(\lambda)S_{,\theta}|^2_{\langle\theta\rangle} - S|_{\langle\theta\rangle}S_{,\theta\theta}|_{\langle\theta\rangle} = 0, \tag{4.9}$$

where

$$g(\lambda) = 2(f^{(1)} - f^{(2)}). \tag{4.10}$$

Eq. (4.9) offers a relation between S and θ through λ in the form shown in (2.27). Note that the differentiations in (4.9) are ordinary (not functional).

For further progress, it is important to make the following observation. Each specific 1pp distribution of S yields a different $\langle S\rangle$, and therefore a different $\langle\theta\rangle$. Consequently, in order for (4.9) to be valid for *any* distribution of S, it must hold for *any* $\langle\theta\rangle$ not only for a specific case. We can therefore generalize (4.9) and write

$$g(\lambda)\cdot S_{,\theta}^2 - S\cdot S_{,\theta\theta} = 0. \tag{4.11}$$

Dividing (4.11) by $SS_{,\theta}$

$$g(\lambda)\frac{S_{,\theta}}{S} - \frac{S_{,\theta\theta}}{S_{,\theta}} = [g(\lambda)\ln(S) - \ln(S_{,\theta})]_{,\theta} = 0. \tag{4.12}$$

Integrating (4.12b) with respect to θ

$$g(\lambda)\ln(S) - \ln(S_{,\theta}) = \ln\left(\frac{S^{g(\lambda)}}{S_{,\theta}}\right) = C \Rightarrow \frac{S_{,\theta}}{S^{g(\lambda)}} = C_1. \tag{4.13}$$

Integrating (4.13) again

$$S(\theta) = [-m(\lambda)(C_1\theta + C_2)]^{-\frac{1}{m(\lambda)}}, \tag{4.14}$$

where

$$m(\lambda) = g(\lambda) - 1 = 2f - 1 = 2(f^{(1)} - f^{(2)}) - 1. \tag{4.15}$$

Before examining C_1 and C_2, note that in order to calculate $\langle P \rangle$ from (2.10), we need $S(\langle\theta\rangle)$ since

$$P_{\langle\theta\rangle} \equiv P_{\langle\theta(S(\theta))\rangle} = \widetilde{P}_{S(\langle\theta\rangle)}. \tag{4.16}$$

This can be found without using C_i as follows. Substitute $\langle\theta\rangle$ in (4.14)

$$S(\langle\theta\rangle) = [-m(\lambda)(C_1\langle\theta\rangle + C_2)]^{-\frac{1}{m(\lambda)}}. \tag{4.17}$$

Then, extract θ from (4.14) and average:

$$\langle\theta\rangle = -\frac{1}{C_1}\left(\frac{\langle S^{-m(\lambda)}\rangle}{m(\lambda)} + C_2\right). \tag{4.18}$$

Extracting $\langle\theta\rangle$ from (4.17) and comparing with (4.18)

$$S(\langle\theta\rangle) = \langle S^{-m(\lambda)}\rangle^{-\frac{1}{m(\lambda)}}. \tag{4.19}$$

Thus, $P(S(\langle\theta\rangle))$ in (2.10) is found without explicitly finding C_i. By (4.3a) and using (4.19) and (4.1), the mean buckling load is obtained by the OFPM:

$$\frac{\langle P \rangle}{\pi^2} = \frac{P|_{\langle\theta\rangle}}{\pi^2} = S(\langle\theta\rangle)^{-1} = \langle S^{-m(\lambda)}\rangle^{\frac{1}{m(\lambda)}} = \langle K^{m(\lambda)}\rangle^{\frac{1}{m(\lambda)}}. \tag{4.20}$$

The above is a realization of the general form in (2.32b).

The power $m(\lambda)$, calculated from (3.9), (3.10) and (4.15) is shown in Fig. 1. The logarithmic scale helps in spanning the whole range of λ. As expected, m is an asymptotic function between -1 and 1, with a "transition" region when λ is in the order of 1.

C_1 and C_2 are also functions of λ, but both are irrelevant for finding $\langle P \rangle$. The reason is that both depend on the boundary conditions of (4.11). However, the LHS of (4.11) is proportional to (2.11) which must vanish. Therefore, C_i can be arbitrary and the solution (4.14) comprises a family of functions.

Nevertheless, C_i can be found in the limit cases. From (3.14), (4.14) and (4.15):

$$m(\lambda \to 0) = -1 \Rightarrow S(\theta) = \theta = C_1\theta + C_2 \tag{4.21}$$

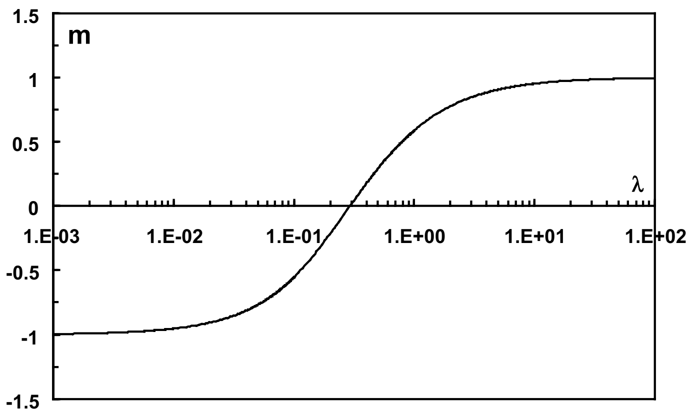

Fig. 1. The power m as a function of grain size λ.

Therefore

$$C_1(\lambda = 0) = 1; \quad C_2(\lambda = 0) = 0. \tag{4.22}$$

Similarly, from (3.15), (4.14) and (4.15):

$$C_1(\lambda \to \infty) = -1; \quad C_2(\lambda \to \infty) = 0. \tag{4.23}$$

This, however, does not affect (4.20).

The coefficient of variation of P_θ, calculated using (2.5) is:

$$\begin{aligned}\rho_{P(\theta)} &= \left(\frac{\mathrm{Var}(P_\theta)}{P_{\langle\theta\rangle}{}^2}\right)^{1/2} = \frac{2\pi^2 S_{,\theta}|_{\langle\theta\rangle}}{S|_{\langle\theta\rangle}{}^2}((w_{x_1}^{(1)})^2 * \langle\theta'_1\theta'_2,\rangle * (w_{x_2}^{(1)})^2)^{\frac{1}{2}} \cdot \left[\frac{\pi^2}{S|_{\langle\theta\rangle}}\right]^{-1} \\ &= \frac{S_{,\theta}|_{\langle\theta\rangle}}{S|_{\langle\theta\rangle}}[\mathrm{Var}(\theta)f^{(1)}(\lambda)]^{1/2}.\end{aligned} \tag{4.24}$$

Using (4.17) and (4.22):

$$\frac{S_{,\theta}|_{\langle\theta\rangle}}{S|_{\langle\theta\rangle}} = C_1\langle S^{-m}\rangle^{-1} = C_1\langle K^m\rangle^{-1} \tag{4.25}$$

and

$$\mathrm{Var}(\theta) = \langle\theta^2\rangle - \langle\theta\rangle^2 = \frac{\langle S^{-2m}\rangle - \langle S^{-m}\rangle^2}{C_1^2 m^2} = \frac{\langle K^{2m}\rangle - \langle K^m\rangle^2}{C_1^2 m^2}. \tag{4.26}$$

Substituting (4.25) and (4.26) in (4.24)

$$\rho_{P(\theta)} = \langle S^{-m}\rangle^{-1}\left[\frac{\langle S^{-2m}\rangle - \langle S^{-m}\rangle^2}{m^2}f^{(1)}\right]^{\frac{1}{2}} = \langle K^m\rangle^{-1}\left[\frac{\langle K^{2m}\rangle - \langle K^m\rangle^2}{m^2}f^{(1)}\right]^{\frac{1}{2}}. \tag{4.27}$$

Thus, $\rho(P)$ is also independent of C_i.

5. Comparison with previous studies

For comparison purposes, it is convenient to follow the data and results considered in Altus and Totry (2003b) and Zhang and Ellingwood (1995). A probability density function $p_K(K)$, which is uniform between two extreme values $(1 - \Delta)$ and $(1 + \Delta)$ is chosen in such a way that $\rho_K = 0.3$, i.e.,

$$p_K = \begin{pmatrix} 1/2\Delta & (1 - \Delta < K < 1 + \Delta) \\ 0 & \text{elsewhere} \end{pmatrix}. \tag{5.1}$$

Therefore,

$$\langle K\rangle = 1; \quad \langle\overline{K}'^2\rangle = \langle\overline{K}^2\rangle - \langle\overline{K}\rangle^2 = \frac{\Delta^2}{3} = 0.09 \to \Delta = 0.5196 \tag{5.2}$$

and

$$\langle K^m\rangle = \int_{1-\Delta}^{1+\Delta}\frac{K^m}{2\Delta}\,\mathrm{d}K = \frac{(1+\Delta)^{1+m} - (1-\Delta)^{1+m}}{2\Delta(1+m)}, \tag{5.3}$$

$$\langle K^{2m}\rangle = \int_{1-\Delta}^{1+\Delta} \frac{K^{2m}}{2\Delta} \mathrm{d}K = \frac{(1+\Delta)^{1+2m} - (1-\Delta)^{1+2m}}{2\Delta(1+2m)}. \tag{5.4}$$

Substituting (5.3) and (5.4) in (4.20) and (4.24)

$$\frac{\langle P\rangle}{\pi^2} = \langle K^m\rangle^{\frac{1}{m}} = \left(\frac{(1+\Delta)^{1+m} - (1-\Delta)^{1+m}}{2\Delta(1+m)}\right)^{\frac{1}{m}}. \tag{5.5}$$

For ρ we obtain

$$\rho_{P(\theta)} = \left[\frac{(1+\Delta)^{1+m} - (1-\Delta)^{1+m}}{2\Delta(1+m)}\right]^{-1} \cdot \left[\frac{(1+\Delta)^{1+2m} - (1-\Delta)^{1+2m}}{2\Delta(1+2m)} - \left(\frac{(1+\Delta)^{1+m} - (1-\Delta)^{1+m}}{2\Delta(1+m)}\right)^2\right]^{\frac{1}{2}} \left(\frac{f^{(1)}}{m^2}\right)^{\frac{1}{2}}. \tag{5.6}$$

Eqs. (5.5) and (5.6) are analytical solutions of $\langle P\rangle$ and ρ_P in terms of the 1pp and 2pp morphology, represented by Δ and λ, respectively.

For comparison, we will also need the associated compliance probability density $p_S(S)$, which is found to be:

$$p_S = \begin{pmatrix} \frac{1}{2\Delta S^2} & ((1-\Delta)^{-1} < S < (1+\Delta)^{-1}) \\ 0 & \text{elsewhere} \end{pmatrix}. \tag{5.7}$$

Once p_S is given, the average and variance of the compliance S are calculated:

$$\langle S\rangle = \int Sp_S \mathrm{d}S = \frac{1}{2\Delta}\mathrm{Ln}\left(\frac{1+\Delta}{1-\Delta}\right) = 1.10814; \quad \langle S^2\rangle = \int S^2 p_S \,\mathrm{d}S = \frac{1}{1-\Delta^2}, \tag{5.8}$$

$$\langle S'^2\rangle = \langle S^2\rangle - \langle S\rangle^2 = \frac{1}{1-\Delta^2} - \left(\frac{1}{2\Delta}\mathrm{Ln}\left(\frac{1+\Delta}{1-\Delta}\right)\right)^2 = 0.1419. \tag{5.9}$$

We now have the necessary morphological information needed to calculate the average and variance of the buckling load on the basis of S (3.7) and K (3.12) by the FPM, or on the basis of θ (4.20, 4.24) by the OFPM. The averages are shown in Fig. 2, and are also compared to Monte Carlo Simulation by Zhang and Ellingwood (1995). All results are normalized to $P_{\langle K\rangle}$, which is exact. For example

$$\frac{\langle P_S\rangle}{P_{\langle K\rangle}} = \frac{\langle P_S\rangle}{P_{\langle S\rangle}}\frac{P_{\langle S\rangle}}{P_{\langle K\rangle}} = \langle \overline{P}_S\rangle\langle K\rangle^{-1}\langle S\rangle^{-1} = 0.902\langle \overline{P}_S\rangle. \tag{5.10}$$

It is seen that while each of the solutions which are based on K or S are accurate only near very small or very large λ, the OFPM is accurate for the whole range. Note that although $P(\langle S\rangle)$ and $P(\langle K\rangle)$ are Ruess and Voigt bounds for *any* λ the 2pp-FPM results are *not* bounds for a *specific* λ.

Fig. 3 shows similar comparisons for the COV(P). The differences between the FPM and the OFPM are more pronounced, but the result is the same: the OFPM matches the MCS much better than previous studies.

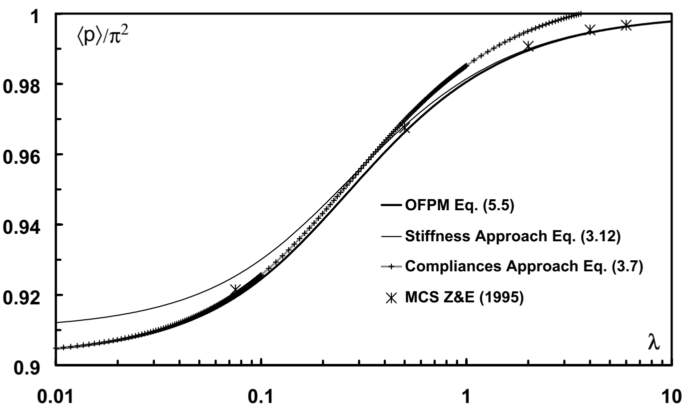

Fig. 2. Average buckling loads as a function of the correlation length λ, calculated by stiffness and compliance perturbation (Eqs. (3.7) and (3.12)), and compared to OFPM (Eq. (5.5)) and FEM–MCS. $\rho_K = 0.3$.

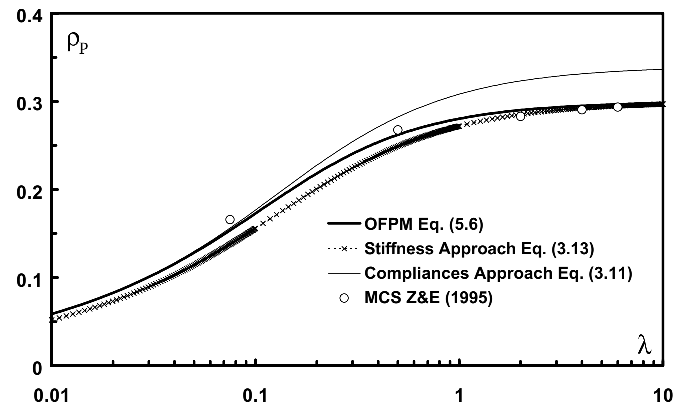

Fig. 3. Coefficient of variation of the buckling load as a function of the correlation length λ, calculated by stiffness and compliance perturbation (Eqs. (3.11) and (3.13)) and compared to the OFPM (Eq. (5.6)). $\rho_K = 0.3$.

6. Discussion and conclusions

The present study deals with two major subjects which are related to stochastically heterogeneous structures, when the typical heterogeneity size is *not* negligible. One is how to find more accurately the average and variance of its macro response in terms of material morphology, and the other is how to define effective material properties for finite size morphologies. A new Optimized Functional Perturbation (OFPM) approach is introduced, based on finding the proper material property, with which to use the FPM in order to eliminate (or minimize) the second order term in its functional series expansion. While current optimization methods of finding effective properties are based on searching for the best value of the homogeneous case (say, modulus S) around which to calculate the heterogeneous case, the OFPM looks for the best *function* $f(S)$. A unique feature of the OFPM is that $f(S)$ depends on both morphology and loading.

A 1D example of buckling of a heterogeneous beam has been solved in details, yielding an analytical solution for $f(S)$ in the form of a power law. It has been shown that while previous methods are capable of producing solutions which are exact at one morphological extreme only, the OFPM is exact at both very small and very large correlation lengths. Therefore, the accuracy at all ranges is improved, as shown by comparing with previous numerical (Finite elements–Monte Carlo Simulations) solutions.

Chapter (2) describes the method for a general 1D heterogeneous medium (elasticity, conductivity etc.,). Therefore, (2.29) is a general governing equation for implementing the OFPM. It is expected that multidimensional cases, where S is a tensor, will yield a similar form, although the practical calculations are more involved. It is not clear, whether the power law form is valid for these cases too, although (2.29) suggests that exploring this subject may be fruitful.

Acknowledgement

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Appendix A

Starting from the functional in Eq. (3.5),

$$\Phi(P) = P^2\Big((S*(w^{(1)})^2)(S*(w^{(2)})^2) - S*(w^{(1)}w^{(2)})^2\Big) - P\frac{\pi^2}{2}\Big(4S*(w^{(1)})^2 + S*(w^{(2)})^2\Big) + \pi^4 = 0. \tag{A.1}$$

The first derivative of (A.1) with respect to θ_1 is:

$$\begin{aligned}\Phi_{,\theta_1} = {} & 2PP_{,\theta_1}\Big[\Big(S*(w^{(1)})^2\Big)\Big(S*(w^{(2)})^2\Big) - \big[S*(w^{(1)}w^{(2)})\big]^2\Big] \\ & + P^2S_{1,\theta_1}\Big[(w_1^{(1)})^2\Big(S*(w^{(2)})^2\Big) + \Big(S*(w^{(1)})^2\Big)(w_1^{(2)})^2 - 2\big[S*(w^{(1)}w^{(2)})\big](w_1^{(1)}w_1^{(2)})\Big] \\ & - \frac{\pi^2}{2}P_{,\theta_1}\Big[4\Big(S*(w^{(1)})^2\Big) + \Big(S*(w^{(2)})^2\Big)\Big] - \frac{\pi^2}{2}PS_{1,\theta_1}\Big[4(w_1^{(1)})^2 + (w_1^{(2)})^2\Big] = 0,\end{aligned} \tag{A.2}$$

where

$$\theta_1 \equiv \theta(x_1); \quad P_{,\theta_1} = \frac{\delta P}{\delta\theta_1}; \quad S_{1,\theta_1} = \frac{\mathrm{d}S(x_1)}{\mathrm{d}\theta_1} \text{ etc.} \tag{A.3}$$

At $\theta_1 = \langle\theta\rangle$ we obtain

$$P_{,\theta_1}|_{\langle\theta\rangle} = -2\pi^2\frac{S_{,\theta}|_{\langle\theta\rangle}}{S\,|_{\langle\theta\rangle}^{\,2}}(w_1^{(1)})^2. \tag{A.4}$$

Note that although the derivative is at $\theta = \langle\theta\rangle$, the above is still a function of x_1.

Similarly, the second derivative of (A.1) with respect to θ_2 is:

$$\begin{aligned}\Phi_{,\theta_1\theta_2} &= 2(P_{\theta_2}P_{,\theta_1} + PP_{,\theta_1\theta_2})\Big[\Big(S*(w^{(1)})^2\Big)\Big(S*(w^{(2)})^2\Big) - \big[S*(w^{(1)}w^{(2)})\big]^2\Big]\\
&+ 2PP_{,\theta_1}S_{2,\theta_2}\Big[(w_2^{(1)})^2(S*(w^{(2)})^2) + \Big(S*(w^{(1)})^2\Big)(w_2^{(2)})^2 - 2\big[S*(w^{(1)}w^{(2)})\big]\Big(w_2^{(1)}w_2^{(2)}\Big)\Big]\\
&+ 2PP_{,\theta_2}S_{1,\theta_1}\Big[\Big(w_1^{(1)}\Big)^2(S*(w^{(2)})^2) + (S*(w^{(1)})^2)\Big(w_1^{(2)}\Big)^2 - 2[S*(w^{(1)}w^{(2)})]\Big(w_1^{(1)}w_1^{(2)}\Big)\Big]\\
&+ P^2S_{1,\theta_1\theta_1}\delta_{12}\Big[(w_1^{(1)})^2(S*(w^{(2)})^2) + (S*(w^{(1)})^2)\Big(w_1^{(2)}\Big)^2 - 2[S*(w^{(1)}w^{(2)})]\Big(w_1^{(1)}w_1^{(2)}\Big)\Big]\\
&+ P^2S_{1,\theta_1}S_{2,\theta_2}\Big[(w_1^{(1)})^2(w_2^{(2)})^2 + (w_2^{(1)})^2(w_1^{(2)})^2 - 2(w_2^{(1)}w_2^{(2)})(w_1^{(1)}w_1^{(2)})\Big]\\
&- \frac{\pi^2}{2}P_{,\theta_1\theta_2}\Big[4(S*(w^{(1)})^2) + (S*(w^{(2)})^2)\Big] - \frac{\pi^2}{2}P_{,\theta_1}S_{2,\theta_2}\Big[4(w_2^{(1)})^2 + (w_2^{(2)})^2\Big]\\
&- \frac{\pi^2}{2}(P_{,\theta_2}S_{1,\theta_1} + PS_{1,\theta_1\theta_1}\delta_{12})\Big[4(w_1^{(1)})^2 + (w_1^{(2)})^2\Big],\end{aligned} \tag{A.5}$$

where

$$P_{,\theta_1\theta_2} = \frac{\delta P}{\delta\theta_1\delta\theta_2}; \quad S_{1,\theta_1\theta_1} = \frac{d^2S_1}{d\theta_1^2} \equiv \frac{d^2S}{d\theta^2}. \tag{A.6}$$

At $\theta_1 = \theta_2 = \langle\theta\rangle$ we obtain

$$\begin{aligned}P_{,\theta_1\theta_2}|_{\langle\theta\rangle} &= \frac{\delta^2P}{\delta\theta_1\delta\theta_2}\bigg|_{\langle\theta\rangle}\\
&= \frac{2\pi^2}{3S|_{\langle\theta\rangle}{}^3}\Big[S_{,\theta}\,|_{\langle\theta\rangle}{}^2\Big(12\Big(w_1^{(1)}\Big)^2\Big(w_2^{(1)}\Big)^2 - 4\Big(w_1^{(1)}w_1^{(2)}\Big)\Big(w_2^{(1)}w_2^{(2)}\Big)\Big) - 3S|_{\langle\theta\rangle}S_{,\theta_1\theta_1}|_{\langle\theta\rangle}\Big(w_1^{(1)}\Big)^2\delta_{12}\Big].\end{aligned} \tag{A.7}$$

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